

# Hausdorff Dimension For Level Sets And $k$ -Multiple Times

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We compute the Hausdorff dimension of the zero-set of an additive Lévy process. As an application, we obtain the Hausdorff dimension formula for the set of  $k$ -multiple times of a Lévy process.

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## 1. INTRODUCTION

Let  $X_{t_1}^1, X_{t_2}^2, \dots, X_{t_N}^N$  be  $N$  independent Lévy processes in  $\mathbb{R}^d$  with their respective Lévy exponents  $\Psi_j$ ,  $j = 1, 2, \dots, N$ . The random field

$$X_t = X_{t_1}^1 + X_{t_2}^2 + \dots + X_{t_N}^N, \quad t = (t_1, t_2, \dots, t_N) \in \mathbb{R}_+^N$$

is called the additive Lévy process. Let  $\lambda_d$  denote Lebesgue measure in  $\mathbb{R}^d$ . Define  $E_1 + E_2 = \{x + y : x \in E_1, y \in E_2\}$  for any two sets  $E_1, E_2$  of  $\mathbb{R}^d$ . The following theorem has recently been proved.

**Theorem 1.1** *Let  $X$  be any additive Lévy process in  $\mathbb{R}^d$  with Lévy exponent  $(\Psi_1, \dots, \Psi_N)$ . Then for any  $F \in \mathcal{B}(\mathbb{R}^d) \setminus \{\emptyset\}$ ,*

$$E\{\lambda_d(X(\mathbb{R}_+^N) + F)\} > 0 \iff \int_{\mathbb{R}^d} |\hat{\mu}(\xi)|^2 \prod_{j=1}^N \operatorname{Re} \left( \frac{1}{1 + \Psi_j(\xi)} \right) d\xi < \infty \quad (1.1)$$

for some probability measure  $\mu$  on  $F$ , where  $\hat{\mu}(\xi) = \int_{\mathbb{R}^d} e^{i\xi \cdot x} \mu(dx)$ ,  $\xi \in \mathbb{R}^d$ .

Khoshnevisan, Xiao and Zhong [1] proved (1.1) under a sector condition. Yang [3] recently removed the sector condition. A special case of Theorem 1.1 is the following theorem.

**Theorem 1.2** *Let  $X$  be any additive Lévy process in  $\mathbb{R}^d$  with Lévy exponent  $(\Psi_1, \dots, \Psi_N)$ . Then*

$$E\{\lambda_d(X(\mathbb{R}_+^N))\} > 0 \iff \int_{\mathbb{R}^d} \prod_{j=1}^N \operatorname{Re} \left( \frac{1}{1 + \Psi_j(\xi)} \right) d\xi < \infty. \quad (1.2)$$

[Note that (1.1) still holds when  $\mathbb{R}_+$  is replaced by  $(0, \infty)$  for some or all components of  $\mathbb{R}_+^N$ .]

Theorem 1.2 is the key for the results of this paper. Meanwhile Theorem 1.1 is also needed in order to relate symmetric stable Lévy processes to stable subordinators. We use a subordination technique to solve the level set problem. To be more precise, we give the formula for Hausdorff dimension of the level set. Subsequently, the intersection time problem and in particular the  $k$ -multiple time problem are solved.

Finally, we mention a  $q$ -potential density criterion: Let  $X$  be an additive Lévy process and assume that  $X$  has an a.e. positive  $q$ -potential density on  $\mathbb{R}^d$ . Then

$$P\left\{F \cap X((0, \infty)^N) \neq \emptyset\right\} > 0 \iff E\left\{\lambda_d(F - X((0, \infty)^N))\right\} > 0. \quad (1.3)$$

The argument is elementary but crucially hinges on the property:  $X_{b+t} - X_b$ ,  $t \in \mathbb{R}_+^N$  (independent of  $X_b$ ) can be replaced by  $X$  for all  $b \in \mathbb{R}_+^N$ ; moreover, the second condition “a.e. positive on  $\mathbb{R}^d$ ” is absolutely necessary for the direction  $\Leftarrow$  in (1.3); see for example Proposition 6.2 of [1].

## 2. LEVEL SETS AND INTERSECTION TIMES

Let  $Z$  be an  $N$ -parameter additive Lévy process in  $\mathbb{R}^d$ . The set  $Z^{-1}(0) = \{t \in (0, \infty)^N : Z_t = 0\}$  is called the level set (at  $0 \in \mathbb{R}^d$ ; otherwise better known as the zero-set). The level set problem, in short, is to compute  $\dim_H Z^{-1}(0)$ .

First, by (1.3) and Theorem 1.2 we obtain immediately that

**Theorem 2.1** *Let  $\{Z; \Psi_1, \dots, \Psi_N\}$  be an  $N$ -parameter additive Lévy process in  $\mathbb{R}^d$ . Assume that  $Z$  has an a.e. positive  $q$ -potential density on  $\mathbb{R}^d$  for some  $q \geq 0$ . Then*

$$P(Z^{-1}(0) \neq \emptyset) > 0 \iff \int_{\mathbb{R}^d} \prod_{j=1}^N \operatorname{Re} \left( \frac{1}{1 + \Psi_j(\xi)} \right) d\xi < \infty. \quad (2.1)$$

Define for  $\alpha \in (0, 1)$  the following  $(N, N)$  additive Lévy process:

$$\sigma_t^\alpha = \sigma_{t_1}^1 + \sigma_{t_2}^2 + \dots + \sigma_{t_N}^N,$$

where

$$\sigma_{t_j}^j = (\sigma_{t_j}^{j,1}, \sigma_{t_j}^{j,2}, \dots, \sigma_{t_j}^{j,N}), \quad j = 1, 2, \dots, N$$

and the  $\sigma_{t_j}^{j,l}$ ,  $1 \leq j \leq N$ ,  $1 \leq l \leq N$ , are  $N^2$  i.i.d. standard  $\alpha$ -stable subordinators with the Laplace exponent  $\lambda^\alpha$ . We call  $\sigma_t^\alpha$  the saturated additive  $\alpha$ -stable subordinator. We note that for each  $t \in (0, \infty)^N$ ,  $\sigma_t^\alpha$  has a density positive everywhere on  $(0, \infty)^N$ . Let  $\Psi_j^\alpha$  be the Lévy exponent of  $\sigma_{t_j}^j$ . A standard  $\alpha$ -stable subordinator with the Laplace exponent  $\lambda^\alpha$  has Lévy exponent

$$|\lambda|^\alpha - i|\lambda|^\alpha \operatorname{sgn}(\lambda)\theta, \quad \lambda \in \mathbb{R},$$

where  $\theta = \tan(\alpha\frac{\pi}{2}) \in (0, \infty)$ . Thus,

$$\Psi_j^\alpha(\xi) = \sum_{l=1}^N |\xi_l|^\alpha - i \left( \sum_{l=1}^N |\xi_l|^\alpha \operatorname{sgn}(\xi_l) \right) \theta, \quad \xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N.$$

**Lemma 2.2** *Let  $X_t$  be a Lévy process in  $\mathbb{R}^d$  with Lévy exponent  $\Psi$  and let  $\sigma^1, \sigma^2, \dots, \sigma^n$  be  $n$  independent subordinators also independent of  $X$ . Then  $X_{\sigma_{t_1}^1 + \sigma_{t_2}^2 + \dots + \sigma_{t_n}^n}$  is an  $n$ -parameter additive*

Lévy process with Lévy exponent  $(\Psi^{\sigma^1}, \Psi^{\sigma^2}, \dots, \Psi^{\sigma^n})$ , where  $\Psi^{\sigma^j}$  denotes the  $\sigma^j$ -subordination of  $\Psi$ .

**Proof** By independence,

$$E e^{i\xi \cdot X_{\sigma_{t_1}^1 + \sigma_{t_2}^2 + \dots + \sigma_{t_n}^n}} = E e^{-(\sigma_{t_1}^1 + \sigma_{t_2}^2 + \dots + \sigma_{t_n}^n)\Psi(\xi)} = \prod_{j=1}^n E e^{-\sigma_{t_j}^j \Psi(\xi)} = \prod_{j=1}^n e^{-t_j \Psi^{\sigma^j}(\xi)}. \quad \square$$

The next lemma follows readily from Lemma 2.2 and the subordination formula.

**Lemma 2.3**  $Z \circ \sigma^\alpha$  is an  $N$ -parameter additive Lévy process in  $\mathbb{R}^d$  (a sum of  $N$  i.i.d. Lévy processes) with Lévy exponent

$$\left\{ \sum_{j=1}^N [\Psi_j(\xi)]^\alpha, \sum_{j=1}^N [\Psi_j(\xi)]^\alpha, \dots, \sum_{j=1}^N [\Psi_j(\xi)]^\alpha \right\}. \quad (2.2)$$

**Lemma 2.4** For any Borel set  $F \subset (0, \infty)^N$ ,

$$P \left\{ F \cap \sigma^\alpha((0, \infty)^N) \neq \emptyset \right\} > 0 \iff E \left\{ \lambda_N(F - \sigma^\alpha((0, \infty)^N)) \right\} > 0. \quad (2.3)$$

**Proof** The direction  $\implies$  is easy since  $\sigma_t^\alpha$  has density. We establish the direction  $\impliedby$ , which is of some interest. Assume that

$$E \left\{ \lambda_N(F - \sigma^\alpha((0, \infty)^N)) \right\} > 0.$$

We extend  $\sigma^\alpha$  to a truly saturated subordinator over the entire time parameter space  $\mathbb{R}^N$ . Due to the special property of the subordinator, which maps  $\mathbb{R}_+$  into  $\mathbb{R}_+$  and thanks to the way we chose to define  $\sigma^\alpha$ , this extension is exactly the one it ought to be. For example, in the second quadrant,

$$\bar{\sigma}^j = (-\bar{\sigma}_{-t_j}^{j,1}, \bar{\sigma}_{t_j}^{j,2}, \dots, \bar{\sigma}_{t_j}^{j,N}), \quad j = 1, 2, \dots, N$$

where  $t_j \in \mathbb{R}_+$ ,  $\bar{\sigma}_{-t_j}^{j,1} \stackrel{d}{=} \sigma_{t_j}^{j,1}$ , and the  $\bar{\sigma}$  are i.i.d. copies of the  $N^2$   $\sigma$ 's in the first quadrant. Then we have a saturated subordinator  $\bar{\sigma}^\alpha = \bar{\sigma}^1 + \dots + \bar{\sigma}^N$  in the second quadrant. After all quadrants have a saturated subordinator, we see that the  $2^N$  saturated subordinators are independent, disjoint in the interior and each is a duplicate of  $\sigma^\alpha$ . Thus, we have defined a process  $\Sigma$  on  $\mathbb{R}^N$ . Since on each open quadrant, the corresponding saturated subordinator has a positive density everywhere for every  $t$  in the same open quadrant, we see that for any  $q > 0$ ,  $\Sigma$  has an a.e. positive  $q$ -potential density on  $\mathbb{R}^N$ . It is also clear that  $\Sigma$  is a process of the property:  $\Sigma_{b+t} - \Sigma_b$ ,  $t \in \mathbb{R}_+^N$  (independent of  $\Sigma_b$ ) can be replaced by  $\sigma^\alpha$  for all  $b \in \mathbb{R}^N$ . Since  $F \subset (0, \infty)^N$ , we can also make each quadrant have a copy of  $F$  through reflection. Let  $F^*$  be the union of the  $2^N$  disjoint copies of  $F$ . Then clearly,

$$P \left\{ F \cap \sigma^\alpha((0, \infty)^N) \neq \emptyset \right\} > 0 \iff P \left\{ F^* \cap \Sigma(\mathbb{R}_I^N) \neq \emptyset \right\} > 0, \quad (2.4)$$

where  $\mathbb{R}_I^N$  is the union of the interiors of the quadrants of  $\mathbb{R}^N$ . What we have done is the so-called “coloring the coordinate axes of the first quadrant”. Since  $F \subset F^*$ ,

$$E \left\{ \lambda_N(F^* - \sigma^\alpha((0, \infty)^N)) \right\} > 0.$$

Since  $\Sigma$  has positive  $q$ -potential density a.e. on  $\mathbb{R}^N$ , and more importantly since  $\Sigma$  has the property:  $\Sigma_{b+t} - \Sigma_b$ ,  $t \in \mathbb{R}_+^N$  (independent of  $\Sigma_b$ ) can be replaced by  $\sigma^\alpha$  for all  $b \in \mathbb{R}^N$ , applying the standard  $q$ -potential density argument shows that

$$P \left\{ F^* \cap \Sigma(\mathbb{R}_I^N) \neq \emptyset \right\} > 0. \quad \square$$

Let

$$S_t^\alpha = S_{t_1}^1 + S_{t_2}^2 + \cdots + S_{t_N}^N$$

be the standard  $N$ -parameter additive  $\alpha$ -stable Lévy process in  $\mathbb{R}^N$  for  $\alpha \in (0, 1)$ ; that is, the  $S^j$  are independent standard  $\alpha$ -stable Lévy processes in  $\mathbb{R}^N$  with the common Lévy exponent  $|\xi|^\alpha$ . A non-trivial result in multi-parameter potential theory is given by the last lemma.

**Lemma 2.5** *For any Borel set  $F \subset (0, \infty)^N$ ,*

$$P \left\{ F \cap S^\alpha((0, \infty)^N) \neq \emptyset \right\} > 0 \iff P \left\{ F \cap \sigma^\alpha((0, \infty)^N) \neq \emptyset \right\} > 0.$$

**Proof** By Lemma 2.4,

$$P \left\{ F \cap \sigma^\alpha((0, \infty)^N) \neq \emptyset \right\} > 0 \iff E \left\{ \lambda_N(F - \sigma^\alpha((0, \infty)^N)) \right\} > 0. \quad (2.5)$$

Since  $S^\alpha$  has an a.e. positive 1-potential density on  $\mathbb{R}^d$ , by (1.3)

$$P \left\{ F \cap S^\alpha((0, \infty)^N) \neq \emptyset \right\} > 0 \iff E \left\{ \lambda_N(F - S^\alpha((0, \infty)^N)) \right\} > 0. \quad (2.6)$$

Let  $\Psi_j^\alpha$  be the Lévy exponent of  $\sigma_{t_j}^j$ . We show that there are two constants  $A_1, A_2 \in (0, \infty)$  such that

$$A_1 \left( \frac{1}{1 + |\xi|^\alpha} \right)^N \leq \prod_{j=1}^N \operatorname{Re} \left( \frac{1}{1 + \Psi_j^\alpha(\xi)} \right) \leq A_2 \left( \frac{1}{1 + |\xi|^\alpha} \right)^N, \quad \forall \xi \in \mathbb{R}^N. \quad (2.7)$$

We have seen that

$$\Psi_j^\alpha(\xi) = \sum_{l=1}^N |\xi_l|^\alpha - i \left( \sum_{l=1}^N |\xi_l|^\alpha \operatorname{sgn}(\xi_l) \right) \theta, \quad \xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N,$$

where  $\theta = \tan(\alpha \frac{\pi}{2}) \in (0, \infty)$ . It follows from simple calculations that

$$\left( \frac{1}{1 + \theta^2} \right) \cdot \frac{1}{1 + \sum_{l=1}^N |\xi_l|^\alpha} < \operatorname{Re} \left( \frac{1}{1 + \Psi_j^\alpha(\xi)} \right) < \frac{1}{1 + \sum_{l=1}^N |\xi_l|^\alpha}.$$

Now, (2.7) follows from the well-known two-sided inequality

$$1 + |\xi|^\alpha \leq 1 + \sum_{l=1}^N |\xi_l|^\alpha \leq C_{\alpha,N}(1 + |\xi|^\alpha)$$

where  $C_{\alpha,N} \in (1, \infty)$  is a constant depending only on  $\alpha$  and  $N$ . Thanks to (2.7), Theorem 1.1 implies that (considering the probability measures on  $-F$ ),

$$E \left\{ \lambda_N(F - S^\alpha((0, \infty)^N)) \right\} > 0 \iff E \left\{ \lambda_N(F - \sigma^\alpha((0, \infty)^N)) \right\} > 0. \quad \square$$

For each  $\beta \in (0, N)$ , choose a  $\sigma^{1-\beta/N}$  independent of  $Z$  and define  $Z \circ \sigma^{1-\beta/N}$ . The following theorem is the main result of this paper.

**Theorem 2.6** *Let  $\{Z; \Psi_1, \dots, \Psi_N\}$  be an  $N$ -parameter additive Lévy process in  $\mathbb{R}^d$ . Assume that for each  $\beta \in (0, N)$ ,  $Z \circ \sigma^{1-\beta/N}$  has an a.e. positive  $q$ -potential density on  $\mathbb{R}^d$  for some  $q \geq 0$ . ( $q$  might depend on  $\beta$ .) [A special case is that if for each  $t \in (0, \infty)^N$ ,  $Z_t$  has an a.e. positive density on  $\mathbb{R}^d$ , then  $Z \circ \sigma^{1-\beta/N}$  has an a.e. positive 1-potential density on  $\mathbb{R}^d$  for all  $\beta \in (0, N)$ .] If  $P(Z^{-1}(0) \neq \emptyset) > 0$ , then almost surely  $\dim_H Z^{-1}(0)$  is a constant on  $\{Z^{-1}(0) \neq \emptyset\}$  and*

$$\dim_H Z^{-1}(0) = \sup \left\{ \beta \in (0, N) : \int_{\mathbb{R}^d} \left[ \operatorname{Re} \left( \frac{1}{1 + \sum_{j=1}^N [\Psi_j(\xi)]^{1-\beta/N}} \right) \right]^N d\xi < \infty \right\}. \quad (2.8)$$

**Proof** According to the argument, Eq. (4.96)-(4.102), in *Proof of Theorem 3.2.* of Khoshnevisan, Shieh, and Xiao [2], it suffices to show that for all  $\beta \in (0, N)$  and  $S^{1-\beta/N}$  independent of  $Z$ ,

$$P \left\{ Z^{-1}(0) \cap S^{1-\beta/N}((0, \infty)^N) \neq \emptyset \right\} > 0 \iff \int_{\mathbb{R}^d} \left[ \operatorname{Re} \left( \frac{1}{1 + \sum_{j=1}^N [\Psi_j(\xi)]^{1-\beta/N}} \right) \right]^N d\xi < \infty. \quad (2.9)$$

By Lemma 2.5,

$$\begin{aligned} P \left\{ Z^{-1}(0) \cap S^{1-\beta/N}((0, \infty)^N) \neq \emptyset \right\} > 0 &\iff P \left\{ Z^{-1}(0) \cap \sigma^{1-\beta/N}((0, \infty)^N) \neq \emptyset \right\} > 0 \\ &\iff P \left\{ 0 \in Z \circ \sigma^{1-\beta/N}((0, \infty)^N) \right\} > 0. \end{aligned}$$

Since  $Z \circ \sigma^{1-\beta/N}$  has an a.e. positive  $q$ -potential density, by (1.3), Lemma 2.3, and Theorem 1.2

$$P \left\{ 0 \in Z \circ \sigma^{1-\beta/N}((0, \infty)^N) \right\} > 0 \iff \int_{\mathbb{R}^d} \left[ \operatorname{Re} \left( \frac{1}{1 + \sum_{j=1}^N [\Psi_j(\xi)]^{1-\beta/N}} \right) \right]^N d\xi < \infty. \quad \square$$

Let  $(X^1; \Psi_1), \dots, (X^k; \Psi_k)$  be  $k$  independent Lévy processes in  $\mathbb{R}^d$  for  $k \geq 2$ . The set

$$T_k = \{(t_1, \dots, t_k) \in (0, \infty)^k : X_{t_1}^1 = \dots = X_{t_k}^k\}$$

is called the intersection time set. Define

$$Z_t = (X_{t_2}^2 - X_{t_1}^1, \dots, X_{t_k}^k - X_{t_{k-1}}^{k-1}).$$

$Z$  is a  $k$ -parameter additive Lévy process taking values in  $\mathbb{R}^{d(k-1)}$ . Clearly,

$$T_k = Z^{-1}(0).$$

Note that for any  $\mathbb{R}^d$ -valued random variable  $X$  and  $\xi_1, \xi_2 \in \mathbb{R}^d$ ,  $e^{i[(\xi_1, \xi_2) \cdot (X, -X)]} = e^{i(\xi_1 - \xi_2) \cdot X}$ . In particular, the Lévy process  $(X^j, -X^j)$  has Lévy exponent  $\Psi_j(\xi_1 - \xi_2)$ . It follows that  $Z^j$ ,  $1 \leq j \leq k$ , has Lévy exponent  $\Psi_j(\xi_j - \xi_{j-1})$  with  $\xi_0 = \xi_k = 0$ . By specializing Theorems 2.1 and 2.6, we have

**Theorem 2.7** *Let  $(X^1; \Psi_1), \dots, (X^k; \Psi_k)$  be  $k$  independent Lévy processes in  $\mathbb{R}^d$  for  $k \geq 2$ . Assume that  $Z$  has an a.e. positive  $q$ -potential density for some  $q \geq 0$ . [A special case is that if for each  $j = 1, \dots, k$ ,  $X^j$  has a one-potential density  $u_j^1 > 0$ ,  $\lambda_d$ -a.e., then  $Z$  has an a.e. positive 1-potential density on  $\mathbb{R}^{d(k-1)}$ .] Then*

$$P(T_k \neq \emptyset) > 0 \iff \int_{\mathbb{R}^{d(k-1)}} \prod_{j=1}^k \operatorname{Re} \left( \frac{1}{1 + \Psi_j(\xi_j - \xi_{j-1})} \right) d\xi_1 \cdots d\xi_{k-1} < \infty \quad (2.10)$$

with  $\xi_0 = \xi_k = 0$ .

Assume that for each  $\beta \in (0, k)$ ,  $Z \circ \sigma^{1-\beta/k}$  has an a.e. positive  $q$ -potential density on  $\mathbb{R}^{d(k-1)}$  for some  $q \geq 0$ , where  $\sigma^{1-\beta/k}$  has  $k$  parameters. ( $q$  might depend on  $\beta$ .) [A special case is that if for each  $t \in (0, \infty)$  and each  $j = 1, \dots, k$ ,  $X_t^j$  has an a.e. positive density on  $\mathbb{R}^d$ , then  $Z \circ \sigma^{1-\beta/k}$  has an a.e. positive 1-potential density on  $\mathbb{R}^{d(k-1)}$  for all  $\beta \in (0, k)$ .] If  $P(T_k \neq \emptyset) > 0$ , then almost surely  $\dim_H T_k$  is a constant on  $\{T_k \neq \emptyset\}$  and

$$\dim_H T_k = \sup \left\{ \beta \in (0, k) : \int_{\mathbb{R}^{d(k-1)}} \left[ \operatorname{Re} \left( \frac{1}{1 + \sum_{j=1}^k [\Psi_j(\xi_j - \xi_{j-1})]^{1-\beta/k}} \right) \right]^k d\xi_1 \cdots d\xi_{k-1} < \infty \right\} \quad (2.11)$$

with  $\xi_0 = \xi_k = 0$ .

Let  $X$  be a Lévy process in  $\mathbb{R}^d$  with Lévy exponent  $\Psi$ . The set for  $k \geq 2$

$$L_k = \{(t_1, \dots, t_k) \in \mathbb{R}_+^k : t_1, \dots, t_k \text{ are distinct, } X_{t_1} = \dots = X_{t_k}\}$$

is called the  $k$ -multiple time set of  $X$ . It is a standard fact that  $L_k$  can be identified with  $T_k$  as long as we replace the  $X^j$  by the i.i.d. copies of  $X$ . Thus, we have found the solution to the multiple-time problem:

**Theorem 2.8** *Let  $(X, \Psi)$  be any Lévy process in  $\mathbb{R}^d$ . Assume that for each  $\beta \in (0, k)$ ,  $Z \circ \sigma^{1-\beta/k}$  has an a.e. positive  $q$ -potential density on  $\mathbb{R}^{d(k-1)}$  for some  $q \geq 0$ . ( $q$  might depend on  $\beta$ .) [A*

special case is that if for each  $t \in (0, \infty)$ ,  $X_t$  has an a.e. positive density on  $\mathbb{R}^d$ , then  $Z \circ \sigma^{1-\beta/k}$  has an a.e. positive 1-potential density on  $\mathbb{R}^{d(k-1)}$  for all  $\beta \in (0, k)$ .] Let  $L_k$  be the  $k$ -multiple-time set of  $X$  for  $k \geq 2$ . If  $P(L_k \neq \emptyset) > 0$ , then almost surely  $\dim_H L_k$  is a constant on  $\{L_k \neq \emptyset\}$  and

$$\dim_H L_k = \sup \left\{ \beta \in (0, k) : \int_{\mathbb{R}^{d(k-1)}} \left[ \operatorname{Re} \left( \frac{1}{1 + \sum_{j=1}^k [\Psi(\xi_j - \xi_{j-1})]^{1-\beta/k}} \right) \right]^k d\xi_1 \cdots d\xi_{k-1} < \infty \right\} \quad (2.12)$$

with  $\xi_0 = \xi_k = 0$ .

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